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Production Sharing Rules and Optimality of Planted-Shared Farming Contracts

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Abstract:

This article examines the optimality problem in existing "planted-shared" agricultural contracts. We define the optimality properties of a long-term contract when there is no agricultural credit market. We use a dynamic principal-agent model with bilateral engagement. This optimal long-term contract highlights two characteristics: first, the agent's remuneration depends on his productive performance; second, the martingale property of the production sharing index highlights his intertemporal smoothing. Moreover, we show that such intertemporal smoothing of the sharing index is a necessary and sufficient condition for the optimality of the long-term agricultural contract. Finally, among the existing contracts, the sharing rules such as the half sharing rate and the third part sharing rate are those that are close to optimal long-term agricultural contracts. Public authorities could promote this type of rules to meet the demand for securing "planted-share" contracts.

Keywords: Farm contract, Principal-agent model, Compensation, Production, Sharing rule.

Classification JEL : D81, D82, Q01, Q15.

1. Introduction

According to Myrdal (1968, cited by Hayami, Ruttan and Malassi, 1998), it is in the agricultural sector that the battle for long-term economic development will be won or lost.The importance of agriculture in the development of an economy is more evident in most African countries where the agricultural sector accounts for a considerable share of gross domestic product (GDP) but where agricultural potential remains largely under-exploited. One way to end such under-exploitation is to easily dispose of abundant land of no immediate value in return for significant income (Paulme, 1962; Léna, 1981; Lesourd, 1982).One way to end this under-exploitation of agricultural potential is to easily give up abundant land that has no immediate use in return for significant income (Paulme, 1962; Léna, 1981; Lesourd, 1982). Such monetarized land transfers make it possible to enhance the value of available land whose exploitation is forced by the lack of family labour force (Chauveau and Richard, 1983). The monetarization of the relationship to land, the needs of land control and the arrival of migrants induce a change in land tenure systems characterized by the transition from the sale to the rental of land. Various contractual forms are used to delegate the land, among which the lease contracts (lease of the land for a fixed price) and sharecropping contracts (lease of the land with an agreement to share production and possibly a fixed sum to be paid) remain the best known. In addition to these two forms of agricultural contracts, other contractual arrangements coexist in a marginal way: - the short-term exchange, land for work (where a land rent is paid in the form of work), - the transfer without specific conditions (fixed-term cultivation contract or open-ended crop contract),- the assignment with explicit consideration (transitional arrangement between free assignment and rental).All these practices concern non perennial crops such as food crops, pineapple and cotton production, etc. (Colin, 2008; Colin and Ruf, 2009).

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The land use with long cycle crops (20 to 40 years): cocoa, coffee, palm, cashew, rubber has allowed the development of another form of contract in African forest areas: the "planted-share" contracts. These agricultural contracts can be defined, in generic terms, as an institutional arrangement where a farmer has access to a long-term right of use, or even to a property right to the land, by showcasing a piece of land through the establishment of a perennial crop and by returning a part of the created plantation to the landowner. This contract is part of a land/labor exchange logic. Several production sharing systems are practiced: in the first moments of the "planted-shared" system, the farmer creates a plantation for the benefit of the land owner. In return, the land owner grants another plot to the farmer to create his own plantation. But the current model is to create a plantation on the same plot and find a sharing rule between the farmer and the landowner. At the time for sharing, it is usually the landowner who chooses the part that belongs to him. The sharing rates most often observed are the following: the half sharing rate over the lifetime of the contract, whether it is plantation sharing or crop sharing; the third part sharing rate over the lifetime of the contract, with one third for the landowner (the transferor) and two thirds for the operator[4] (the farmer). Other sharing rates combine the two models above: - one third for the transferor/two thirds for the operator in the first period and half and half in the second period,- one hundred per cent for the operator in the first period and half and half in the second period" (Colin, 2008). The four sharing rules above can be summarized into two groups: the first two rules are used to smooth the sharing rate all over the contract, and the last two rules do not smooth the sharing rate over the life of the contract. These production sharing rules lead to a high potential conflict between "planted-shared" contracts. Enforcement routinely challenges the rights of the transferor or operator, leading to conflict and tension.

This explains the great demand to safeguard these contracts through public validation of sharing rules. It is therefore essential to foster the growth of a better organized land market in rural areas by making available appropriate legal and contractual tools relating to the leasing of rural land. This can enhance the relationship between land rights owners and non-owners and ensure the prevention and settlement of rural land conflicts.

In this paper, the aim is to theoretically analyze the optimality of a "planted-shared" contract that satisfies the contracting parties while resolving the potential conflicts and we compare the existing contracts with that optimal contract. To do this, we use a dynamic contract model with bilateral commitment using the principal-agent paradigm. This paradigm is based on both a spot agricultural contract model from Dubois (1999) and a dynamic model from Laffont and Martimort (2002) which deals with non-agricultural contracts.

The article is organized as follows: after reviewing the economic literature on agricultural contracts (section 2), we use the principal-agent paradigm to define the characteristics of an optimal long-term contract model with incentives and bilateral commitments (section 3). Finally we discuss the results, make recommendations and conclude the article (section 4).

2. Literature review

Early economic work on agricultural contracts attempted to explain the choice of agricultural contracts by invoking reasons related to risk sharing, transaction costs or various types of constraints (financial constraints, labour constraints, etc.). The first studies on agricultural contracts focus on delegation and land use agreements between landowners and farmers, especially sharecropping and tenant farming. The theoretical study of contracts is based on transaction cost theory (Williamson, 1971) and risk and information theory (Laffont and Tirole, 1986; Allen and Lueck, 1995). The three broad categories of transaction costs (research and information costs, negotiation costs, policing and execution costs) explain contract choices theoretically and empirically. The risk and information theory approach studies contracts between farmers and firms using the principal-agent paradigm. These contracts are understood as a way for the firm to delegate responsibilities to the farmer while managing risk sharing arrangements and providing the right incentives. Cheung (1969) explains the choice of contract by a tradeoff between risk sharing and transaction cost minimization. He is the first to explicitly use the concept of transaction costs in the analysis of agrarian contracts. In line with this approach, models consider tenant farming as a contract that makes it possible, with some conditions, to reduce transaction costs compared to tenant farming and direct tenure with salaried labour (Allen and Lueck, 1995, 1999).

[4]The farmer is in a symmetrical relationship to that of the "aboussan" manoeuvre: while the "aboussan" manoeuvre is remunerated at a third part of the production, the farmer of a "shared planted" contract receives two thirds of the production

Datta et al (1986) explain the choice of agricultural contract through an evaluation of the costs paid by the landowner because of the risks of opportunistic behaviour on the part of the tenant: - risk of over-exploitation of the land resource (case of tenant farming), - risk of less effort in the work of the laborer (case of the direct worker with salaried employment), - opportunistic behaviour of the tenant-holder concerning his investment in labour, land mining, fraud in the sharing, etc.

Other models, using the principal-agent approach, give a primary role, not to transaction costs but to risk in explaining contractual choices. The pioneering work is Stiglitz (1974). Because of the problem of moral hazard generated by the non-observability of the agent's actions, the optimal tenant farming contract results from a trade-off between incentives and risk sharing for a risk neutral landowner and a risk adverse farmer.In addition to transaction costs and risk sharing, the studies provide other reasons for the choice of agrarian contract. Eswaran and Kotwal (1985) explain the choice of tenant farming who may skillfully achieve off-market coordination of the benefits related to cost sharing and the complementarities of resources provided by each partner. In their model, landowners have better technical and economic management capacity, while tenant farming have better capacity to supervise family work and reduce opportunistic behaviours in pooling the labour force. The interest of these two actors regarding the production relates to the strong decrease of the moral hazard risk, which is no longer limited to only the investment in the owner's work but integrates other risks: land degradation, fraud in product sharing, etc.

According to Laffont and Matoussi (1995), the choice of the type of contract (rental, half, third or quarter) is determined by an increasing financial constraints assigned to the tenant, with a simultaneous increasing non-incentive effect, however mitigated by a logic of repeated games. According to Shetty (1988), contract choice depends on the assets held by the tenants. In this case, the landowner will not delegate the land use to a tenant with a low level of capital accumulation, as no security can be provided in the case of default in rent payments. These producers are therefore forced to enter into sharecropping contracts, while richer tenants enter tenant farming contracts and will generate higher returns.

Agricultural contracts are also a substitute for the credit market. This is the case of sharecropping/credit contracts where the landowner grants credit to the tenant, which is in effect secured by production on the land that is the subject of the sharecropping contract (Hayami and Otsuka, 1993 ; Jaynes, 1982). This type of contract induces behaviours that lead the landowner to urge the tenant going into debt with him because, he can change the terms of the loan contract in order to force the tenant to more work (Braverman and Stiglitz, 1982). In addition, in the context of a non-monetarized economy, the tied contract may reduce implementation costs (Bardhan, 1991). As we see, the forms of agricultural contracts arise as an optimal response to the occurrence of risks, transaction costs and imperfections of the capital market (financial constraints) and the rural labour market (lack of labour). The imperfections of the insurance market may also explain the interest of sharecropping for both the tenant and the landowner. Sharecropping also works as a substitute for the insurance market. Households accessing tenant farming contracts that share production risks are better insured than others (Dubois, 2000). Thus, these contracts allow households to better insure themselves against risks. They are used to supplement markets by providing risk-averse households with contingent assets that no other combination of accessible assets can provide.

In recent years, new types of agricultural contracts have emerged. In the first ones, called contract farming, we have some production and/or marketing agreements between an agribusiness firm and a farmer. The others are called landlord-tenant contracts. They are land use delegation agreements existing between a landowner and a farmer. In the first type of agreement, a farmer agrees to deliver specified quantities of agricultural products at the required quality standards and at a specified time to a firm that undertakes to purchase them at a predetermined price as well as providing inputs and technical support for production. As in the previous case, the theoretical study of these contracts is based on the transaction cost theory and the risk and information theories. In the transaction cost approach, contracts between farmers and agribusiness firms, both risk neutral, are similar to vertical coordination in the agricultural supply chain and can be analyzed using vertical integration tools (Hennessy, 1996; Leathers, 1999). This work analyses the reasons justifying the adoption of such an industrial organization and the benefits brought to the parties engaged in the contract. The risk and information economics approach uses contract theory tools to determine optimal contracts when informational rents have a significant role. Different types of contracts are studied, including those in the tomato sector (Alexander et al, 2000; Hueth and Ligon, 2002) and the chicken industry (Knoeber and Thuman, 1995; Goodhue, 2000).

Glover and al (1994) show that pre-determined contract prices between farmers and an agribusiness firm confront stakeholders with the market spot price volatility, which induces optimal risk sharing and improves production efficiency. Other works analyze contracts in the agricultural supply chain when one party does not have perfect information on one or more characteristics of the other party (Knoeber and Thuman, 1995; Goodhue, 2000; Hueth and Ligon, 2002). In addition to optimal risk allocation, increased production efficiency and lower information rents, agricultural contracts have other advantages. They often stabilize the supply chain of perishable products when the number of potential purchasers and sellers is low (MacDonald and Korb, 2011). They also help promote new agricultural products and new technologies (Boehlje et al, 1998).

All the above mentioned studies have two basic characteristics: firstly, agricultural contracts are mainly straight-line contracts; secondly, agricultural contracts are analyzed as spot contracts, i.e. static contracts. Concerning the straight linearity of agricultural contracts, the studies are based on the need to be as close as possible to the real world (Stiglitz, 1974; Eswaran and Kotwal, 1985; Laffont and Matoussi, 1995; Dubois, 1999). The straight-line sharing rule between the landowner and the tenant involves the landowner offering the tenant a fixed portion of the production. However, generally, straight-line contracts are not perceived as optimal. The linear sharing rule is less powerful than the optimal second-best contract. For example, Laffont and Matoussi (1995) show that the straight-line contract, even if it provides sufficient incentives for effort, is an inefficient way of extracting agents' rents. The optimal second rank contract shows that the agent's production can have a zero return. However, the return is positive with the straight-line sharing rule. Thus, with a straight-line sharing rule, it is difficult to punish agents for poor performance. Agents always have a strictly positive informational rent. However, with some assumptions about contractual ability and preferential treatment, it is possible to pinpoint the optimum linearity of the agricultural contract (Holmstrom and Milgrom, 1987).

Concerning the nature of agricultural contracts as spot contracts, it should be stressed that even when the relationship between the principal and the agent is intended to last for a long time, it is only a question of replicating spot contracts. Such repetition of the static optimal contract is never optimal (Rogerson, 1985). But the importance of such static contractual forms is due to an inability to make long-term credible commitments (Dubois, 1999). However, with "planted-shared" contracts mainly practiced in forest areas in Africa, we have an example of agricultural contracts that are not static spot contracts. These "planted-shared" contracts are dynamic agricultural contracts that can be analyzed within the framework of the theory of long-term contracts in the principal-agent paradigm with bilateral commitment of the actors. Long-term contracts have been studied in contexts other than agriculture. In these studies, the optimal long-term contract is usually a memory contract that smoothes the agent's income while transferring a part of the payment to the second period, which requires a commitment from the principal (Lambert, 1983; Rogerson, 1985a; Chiappori et al, 1994). But, sometimes the optimal long-term contract is a contract without memory where the agent's second period remuneration only depends on the second period. Formally, if τ^{ij} is the second period return accordind to a second period performance $j$and a first period performance i, we have $\tau^{ij} = \tau^{kj}$ for all, $k = 1, \dots, n$. The "planted-shared" contracts specifically related to perennial crops are analyzed here as long-term straight-line contracts. In the following, we will attempt to understand the characteristics of such long-term straight-line contracts. The challenge is to ensure and prevent rural land conflicts, as has been pointed out.

3. The principal-agent model

3.1. Model Assumptions

We consider a principal-agent relationship that takes place over two periods. During each period the agent (operator) chooses a level of effort, and then the nature determines the resulting output: the level of production. The production $y_t$is observable and contestable. Effort e_t is unobservable by the principal (the landholder). Monitoring costs are assumed to be too high. Suppose that the effort e can only take two values in $\{0,1\}$. The costs of effort are rated $C(1) = C$ and $C(0) = 0$. At each period, the effort of the agent allows to reach a stochastic output $\tilde{y}_t = \overline{y}$ (respectively$\underline{y}$) with probability $P^i(e_t)$ (respectively$(1 - P^i(e_t))$. We notice $P_1^i = P^i(1)$ and $P_0^i = P^i(0)$and $\Delta P^i = P_1^i - P_0^i$. Thus, $P^i(e)$ is the probability of obtaining the result i when the level of effort is e. These probabilities are assumed to be the same for both periods. Outputs are assumed to be independent from period to period. Payment for each period is contingent on past and present output.

In the first period, the agent receivesτ^iif the observed result is i. Similarly, in the second period, he receives the remuneration τ^{ij} if he obtained the result i in the first period then the result j in the second period. We assume that there is no agricultural credit market. We note a, the sharing rate between the principal and the agent and we consider straight-line contracts of the form:

$$\tau^i = a^i y_1 + b \quad \text{with } y_1 \in \left\{\underline{y}_1, \overline{y}_1\right\} \; and \; \tau^i \in \left\{\underline{\tau}^i, \overline{\tau}^i\right\} \quad (1)$$

$$\tau^{ij} = a^{ij} y_2 + b \quad \text{with } y_2 \in \left\{\underline{y}_2, \overline{y}_2\right\} \text{and } \tau^{ij} \in \left\{\underline{\tau}^{ij}, \overline{\tau}^{ij}\right\} \quad (2)$$

The agreement relationship between the principal and the agent shall be devoted to the payment of the remuneration at the end of each period. But, payments are conditioned by available output information: first period output for first payment and both outputs for the second payment. Neither the agent nor the principal can leave the contract. Both contracting parties can enter into a long-term contract because it is a bilateral contract. We assume no discounting; this simplifies the calculations without changing the conclusions. The principal is assumed to be risk neutral, and its objective function is written as follows:

$$V = P_1^i\left(\overline{y}_1 - \overline{\tau}^i + P_1^i\left(\overline{y}_2 - \overline{\tau}^{ij}(\overline{y}_2)\right) + \left(1-P_1^i\right)\left(\underline{y}_2 - \underline{\tau}^{ij}(\overline{y}_2)\right)\right)$$
$$+ \left(1-P_1^i\right)\left(\underline{y}_1 - \underline{\tau}^i + P_1^i\left(\overline{y}_2 - \overline{\tau}^{ij}(\underline{y}_2)\right) + \left(1-P_1^i\right)\left(\underline{y}_2 - \underline{\tau}^{ij}\left(\underline{y}_2\right)\right)\right) \quad (3)$$

It is assumed the agent is risk averse. Without that assumption, the moral hazard problem seems rather trivial. The Von Neumann-Morgenstern utility function $u(.)$ of the agent is therefore strictly growing and concave.In addition, it is assumed that the range of possible values of the payment and of the utility is not bounded. Then, the agent's utility function is:

$$U(\tau_1(y_1), \tau_2(y_2), e_1, e_2) = u(\tau_1(y_1)) - C(e_1) + u(\tau_2(y_2)) - C(e_2).$$

In stationary framework analysis, the separability properties of the utility function and the independence conditions of the probability distributions are high enough to avoid introducing an adverse selection dimension and maintain a pure moral hazard model. The principal decides to make concrete both the effort e_1 in the first period and the effort e_2^i in the second period if the result i has been observed in the past In that case, the principal seeks to minimizing the costs that will induce the agent to engaging in such behaviour. For any result i obtained previously, the agent's incentive constraint in the second period is:

$$P_1^i u_2\left(\overline{\tau}^{ij}(y_2)\right) + \left(1-P_1^i\right)u_2\left(\underline{\tau}^{ij}(y_2)\right) - C \geq P_0^i u_2\left(\overline{\tau}^{ij}(y_2)\right) + \left(1-P_0^i\right)u_2\left(\underline{\tau}^{ij}(y_2)\right) \quad (4)$$

Let's assume the followings equations:

$u(\tau^i(y_1)) = u_1$; $u(\tau^i(\overline{y}_1)) = \overline{u}_1$; $u\left(\tau^i(\underline{y}_1)\right) = \underline{u}_1$; and $u(\tau^{ij}(y_2)) = u_2(y_1)$; $u_2\left(\overline{\tau}^{ij}(y_2)\right) = \overline{u}_2(y_1)$; $u_2\left(\underline{\tau}^{ij}(y_2)\right) = \underline{u}_2(y_1)$,

The above constraint is written:

$$P_1^i\overline{u}_2(y_1) + \left(1-P_1^i\right)\underline{u}_2(y_1) - C \geq P_0^i\overline{u}_2(y_1) + \left(1-P_0^i\right)\underline{u}_2(y_1) \Rightarrow \overline{u}_2(y_1) - \underline{u}_2(y_1) \geq \frac{C}{\Delta P^i} \quad (5)$$

Similarly, in the first period, we consider fulfilled the second period constraint. So, the agent's incentive constraint is written as follows:

$$\overline{u}_1 + P_1^i\overline{u}_2(\overline{y}) + \left(1-P_1^i\right)\underline{u}_2(\overline{y}) - \left(\underline{u}_1 + \left(P_1^i\overline{u}_2\left(\underline{y}\right) + \left(1-P_1^i\right)\underline{u}_2\left(\underline{y}\right)\right)\right) \geq \frac{C}{\Delta P^i} \quad (6)$$

In the class of contracts that deal with such the inequalities, the agent's second period participation constraint is written, for any result i obtained in the past:

$$P_1^i\overline{u}_2(y_1) + \left(1-P_1^i\right)\underline{u}_2(y_1) - C \geq u_2(y_1) \quad (7)$$

This constraint reveals that the agent is not committed. Imposing the verification of this constraint implies that the agent has not entered a long-term relationship. Therefore, the bilateral contract hypothesis requires the removal of this constraint.

The intertemporal participation constraint is:

$$P_1^i\left(\bar{u}_1+\left(P_1^i\bar{u}_2(\bar{y})+(1-P_1^i)\underline{u}_2(\bar{y})\right)\right)+(1-P_1^i)\left(\underline{u}_1+\left(P_1^i\bar{u}_2\left(\underline{y}\right)+(1-P_1^i)\underline{u}_2\left(\underline{y}\right)\right)\right)-2C\geq 0 \quad (8)$$

The principal aims at designing a long-term contract (τ^i,τ^{ij}) that maximize his expected utility, and provide the right incentives for the agent's effort. This contract must guaranty the proper risk-sharing and the best upkeep of the land. We consider$h$ is the inverse function of the agent's utility function and we consider the following equations:

$\tau^i(\bar{y}_1)=\bar{\tau}^i=u^{-1}(\bar{u}_1)=h(\bar{u}_1)$; $\underline{\tau}^i=h(\underline{u}_1)$; $\bar{\tau}^{ij}(y_2)=h(\bar{u}_2(y_1))$;

$$\underline{\tau}^{ij}(y_2)=h(\underline{u}_2(y_1)).$$

The program of the principal is to maximise the objective-function V under participation and incentive constraints. This program is then written as follows:

$$\begin{aligned}Max_{\{(\bar{u}_1,\underline{u}_1)(u_2(\bar{y}),u_2(\underline{y}))\}}V\\ &=P_1^i\left(\bar{y}_1-h(\bar{u}_1)+P_1^i\left(\bar{y}_1-h(\bar{u}_2(\bar{y}))\right)+(1-P_1^i)\left(\underline{y}_1-h(\underline{u}_2(\bar{y}))\right)\right)\\ &+(1-P_1^i)\left(\underline{y}_1-h(\underline{u}_1)+P_1^i\left(\bar{y}_1-h\left(\bar{u}_2(\underline{y})\right)\right)+(1-P_1^i)\left(\underline{y}_1-h\left(\underline{u}_2(\underline{y})\right)\right)\right)\end{aligned}$$

The constraints are equation (5), (6)and (8)

3.2. Program resolution

To resolve the optimization program P, a two-step process is followed.

The first step consists in optimizing the principal's pay-off in the second period in order to determine its continuation value of the contract$V_2(u_2(y_1))$. That value is basically the value of the program $P_2(y_1)$ which optimizes the following objective-function:

$$Max_{\{\bar{u}_2(y_1),\underline{u}_2(y_1)\}}P_1^i\left(\bar{y}_2-h(\bar{u}_2(y_1))\right)+(1-P_1^i)\left(\underline{y}_2-h(\underline{u}_2(y_1))\right) \quad (9)$$

The constraints (5), (6) and (8)are saturated to optimum and the following optimal solutions are obtained:

$\bar{u}_2^D(y_1)=C+u_2(y_1)+(1-P_1^i)\frac{C}{\Delta P^i}\Rightarrow \underline{u}_2^D(y_1)=C+u_2(y_1)-P_1^i\frac{C}{\Delta P^i}$.

The second rank cost to implement high effort in period 2 in accordance with the promise of second period utility$u_2(y_1)$is:

$C_2^{SB}(u_2(y_1))=P_1^ih\left(C+u_2(y_1)+(1-P_1^i)\frac{C}{\Delta P^i}\right)+(1-P_1^i)h\left(C+u_2(y_1)-P_1^i\frac{C}{\Delta P^i}\right)\Rightarrow C_2^{SB}(u_2(y_1))=$
$P_1^ih\left(\bar{u}_2^D(y_1)\right)+(1-P_1^i)h\left(\underline{u}_2^D(y_1)\right)$.

Thus, we have as continuation value of the contract for the principal:

$V_2(u_2(y_1))=P_1^i\bar{y}_2+(1-P_1^i)\underline{y}_2-C_2^{SB}(u_2(y_1))$.

The second step consists in going back to the initial program P resolution by rewriting it in the form of the next program P':

$$\begin{cases}Max_{\{(\bar{u}_1,\underline{u}_1)(u_2(\bar{y}),u_2(\underline{y}))\}}P_1^i\left(\bar{y}_1-h(\bar{u}_1)\right)+(1-P_1^i)\left(\underline{y}_1-h(\underline{u}_1)\right)+\left(P_1^iV_2(u_2(\bar{y}))+(1-P_1^i)V_2\left(u_2\left(\underline{y}\right)\right)\right)\\ \qquad s.c\ \ \bar{u}_1+u_2(\bar{y})+\left(\underline{u}_1+u_2\left(\underline{y}\right)\right)\geq\frac{C}{\Delta P^i}\\ P_1^i(\bar{u}_1+u_2(\bar{y}))+(1-P_1^i)\left(\underline{u}_1+u_2\left(\underline{y}\right)\right)\geq C\end{cases}$$

The two constraints of this new program $P'$are the new writings of constraint (5)and (6).Consider the multipliers λ and μ of these constraints. As the P' program is a concave problem with linear constraints, the first order conditions of Kuhn and Tucker are necessary and sufficient to characterize optimality:

$$\begin{cases} P_1^i h'(\bar{u}_1^D) = \lambda + \mu P_1^i & (10) \\ (1-P_1^i)h'(\underline{u}_1^D) = -\lambda + \mu(1-P_1^i) & (11) \\ P_1^i C_2^{SB'}\left(u_2^D(\bar{y})\right) = \lambda + \mu P_1^i & (12) \\ (1-P_1^i)C_2^{SB'}\left(u_2^D(\underline{y})\right) = -\lambda + \mu(1-P_1^i) & (13) \end{cases}$$

The resolution of this system of equations yields the following pairs of results: $(\bar{u}_1^D, \underline{u}_1^D)$ and $\left(u_2^D(\bar{y}), u_2^D(\underline{y})\right)$ *(proof in appendix 1):*

$$h'(\bar{u}_1^D) = P_1^i h'\left(\bar{u}_2^D(\bar{y})\right) + (1-P_1^i)h'\left(\underline{u}_2^D(\bar{y})\right) \qquad (14)$$

$$h'(\underline{u}_1^D) = P_1^i h'\left(\bar{u}_2^D(\underline{y})\right) + (1-P_1^i)h'\left(\underline{u}_2^D(\underline{y})\right) \qquad (15)$$

By noting $E_{\tilde{y}_2}(.)$ = the expected operator relative to the distribution of the second period output induced by a high effort in the same period, and $\tilde{u}_2^D$ = the random variable of the second period utilities, the two equations above satisfy the martingale property and thus become simplified as follows :

$$h'(u_1^D(y_1)) = E_{\tilde{y}_2}\left(h'(\tilde{u}_2^D(y_1))\right) \text{for all } y_1 \in \left\{\underline{y}, \bar{y}\right\} \qquad (16)$$

The martingale property reflects the tendency of a random variable to remain centered around its previous value. Remembering that $h(u_t^D(y_t))$ is the agent's remuneration in t, we have the marginal payment per additional unit of production noted $h'(u_t^D(y_t))$. The martingale property given by equation (16) is interpreted as follows: The optimal contract that motivates the agent to make more effort, is a contract that smoothes the agent's income intertemporally. Thus, a high first-period effort is not fully paid in the first period; the payment is spread out over the second period. Such smoothing of remuneration is compatible with the assumption that there is no agricultural credit market. Further fine-tuning of equation (16) makes it possible to highlight the smoothing of the production sharing index. To see it, let's first remember that

$h(u_1^D(y_1)) = \tau^i(y_1) = a^i y_1 + b$ and $h\left(\tilde{u}_2^D(y_1)\right) = \tilde{\tau}^{ij}(y_2) = \tilde{a}^{ij} y_2 + b$ *(linear contract assumption)*

In addition, we consider the following classifications: $A^i = \frac{1}{a^i}$; $A^{ij} = \frac{1}{\tilde{a}^{ij}}$. If a^i and $\tilde{a}^{ij}$ are sharing-rates, then A^i and A^{ij} are the inverses of the sharing-rates that we will call sharing-indexes. We can then show that we can write a martingale solution on production sharing indices in the following way *(proof in appendix 2):*

$$A^i = M \times E_{\tilde{y}_2}(A^{ij}) \text{where } M \equiv E_{\tilde{y}_2}\left[\frac{h_y'\left(\tilde{u}_2^D(y_1)\right)}{h_y'(u_1^D(y_1))}\right] \qquad (17)$$

with $M \gtrless 1$, we have :

$$A^i \gtrless E_{\tilde{y}_2}(A^{ij}) \qquad (18)$$

If $M = 1$, we have $A^i = E_{\tilde{y}_2}(A^{ij})$: the optimal sharing-index is a martingale.
If $M > 1$, we have $A^i > E_{\tilde{y}_2}(A^{ij})$: the optimal sharing-index is an over martingale.
If $M < 1$, we have $A^i < E_{\tilde{y}_2}(A^{ij})$: the optimal sharing-index is a sub-martingale.

3.3. Findings interpretation

Equation (17) shows that the optimal sharing index obtained at the end of each crop cycle is a martingale; it depends on the past history (memory) of the relationships. That reflects the intertemporal smoothing effect of the agent's income, in the absence of credit market. The optimal contract "planted-shared", a long-term contract, highlights a memory effect linked to the consideration of present and past performances in payment terms. The optimal "planted-shared" contract, a long-term contract, highlights a memory effect resulting from the present and past performance taken into account in payment terms. Indeed, we observe that the second period sharing-index depends on the first period sharing-index. $A^{ij} \neq A^{kj}$ for all i and $k = 1, \ldots, n$.

Moreover, the first period sharing-index $A^i$is not only based on the i result of the same period but also depends on the j results and the A^{ij} sharing-index of all other periods. The results are summarized in the following proposition:

Proposition 1: Based on the assumptions of no agricultural credit market, the optimal "planted-shared" type contract is a long-term memory contract. Specifically, the optimal income sharing rate from agricultural production is a martingale:$A^i = E_{\tilde{y}_2}(A^{ij})$, which reflects an intertemporal smoothing effect of the agent's income.

This result can be intuitively explained as follows: a first period strong performance reflects, with a given effort, favorable random circumstances. The corresponding additional gain, necessary to ensure a correct incentive effect, is transitory. There is no reason for it occurs anymore in the future. The agent will therefore seek to spread this gain over time. That is only possible in the context of a memory contract[5]. If the agent achieves a good performance in the first period, corresponding to a high payment[6], the optimal long-term contract will then smooth out his income by transferring a part of the payment to the second period. This requires a commitment from the principal. This result obtained in the context of a long-term agricultural contract is similar to that obtained for other types of contract, (Lambert, 1983 ; Rogerson, 1985a ; Chiappori et al, 1994).

To see it formally, we start from equation(17):

$$A^i = E_{\tilde{y}_2}(A^{ij})E_{\tilde{y}_2}\left[\frac{h_y'\left(\tilde{u}_2^D(y_1)\right)}{h_y'(u_1^D(y_1))}\right].$$

Due to the smoothing of the remuneration, we have

$$\frac{h_y'\left(\tilde{u}_2^D(y_1)\right)}{h_y'(u_1^D(y_1))} \approx 1 \Rightarrow A^i \approx E_{\tilde{y}_2}(A^{ij}).$$

The agent smoothes the sharing-index to the optimum, i.e. the index is a martingale ($A^i \approx E_{\tilde{y}_2}(A^{ij})$).

Let us see what happens when the agent does not smooth the production sharing-index. It comes down to analyzing the implications of an index that is not a martingale. Are we still at the optimum in this case? First, let's assume that the sharing-index is an over-martingale (where$A^i > E_{\tilde{y}_2}(A^{ij})$). Due to the smoothing of the remuneration, we have$\frac{h_y'\left(\tilde{u}_2^D(y_1)\right)}{h_y'(u_1^D(y_1))} > 1$. The agent is better paid in the second period than in the first period if he makes an extra effort in each period. The long-term intertemporal-based contract does not stimulate effort.

When the sharing index is a sub-martingale (where $A^i < E_{\tilde{y}_2}(A^{ij})$),

we have $\frac{h_y'\left(\tilde{u}_2^D(y_1)\right)}{h_y'(u_1^D(y_1))} < 1$:The agent is better paid in the second period than in the first period if he makes an extra effort in each period. Once again, the long-term intertemporal-based contract does not stimulate effort. It implies the following proposition:

Proposition 2: The martingal property of the production sharing index is not only a necessary but also a sufficient condition for the optimality of the long-term "planted-shared" type agricultural contract. The necessary and sufficient condition for the optimality of this contract is the intertemporal smoothing of the sharing index.

We can otherwise check that the intertemporal smoothing solution of the agent's income is optimal for the principal. To do this, we see whether, because of the agent's risk aversion, a strong performance in the first period is fully rewarded or not in the first period, to the best (optimally). It was noted that:

$u(\tau^i) = u(a^i y_1 + b), u(\tau^{ij}) = u(a^{ij} y_2 + b), u(a^i y_1 + b) = u_1(y)$ et $u(a^{ij} y_2 + b) = u_2(y)$.

If the agent is fully rewarded in the first half, at the optimum for high performance, this means that he is not rewarded for high performance in period 2:$u_2(\overline{y}) = u_2(\underline{y})$.

If the agent is not fully rewarded in the first period, at the optimum for high performance, it means a part of the first period reward is shifted to the second period:, $u_2(\overline{y}) > u_2(\underline{y})$.

[5] This logic is however particular to the case of a stationary reservation utility.

[6] The marginal utility of the payment must be lower than the harmonic mean of the marginal utilities in the static contract.

If the agent is rewarded in the first period for high performance, this implies a greater utility for the high level of production, i.e. formally: $u_1(\overline{y}) > u_1(\underline{y})$.

With an absurd reasoning, one will assume that the agent is not rewarded for a high performance in period 2; therefore, $u_2(\overline{y}) = u_2(\underline{y})$,which implies that $u_2(y)$is a constant function.Suppose $F$is that function. We have:$u_2(y) = u(\tau^{ij}) = F \Rightarrow \tau^{ij} = u^{-1}(F)$.

This equation is an example of how second-period remuneration does not consider either first-period performance or second-period performance. In such a case, the agent would not make an effort in the second period, in opposition to the optimal incentive situation. Consequently, one cannot optimally have$u_1(\overline{y}) > u_1\left(\underline{y}\right) \Rightarrow u_2(\overline{y}) = u_2\left(\underline{y}\right)$;

It necessarily turns out that$u_1(\overline{y}) > u_1\left(\underline{y}\right) \Rightarrow u_2(\overline{y}) > u_2\left(\underline{y}\right)$, It means a part of the first period reward is postponed to the second period, because the agent is not fully optimally rewarded in the first period. The following proposition summarizes this result:

Proposition 3: Giving in the first period, the entire first-period reward is, intertemporal-based sub-optimal. The principal will still prefer smoothing the reward in order to minimize costly implementation of a high effort in period 1.

Let's remember that the goal is to compare existing contracts with the theoretically optimal long-term contract. These existing contracts are differently designed, in terms of sharing rules, with the regard of the lifetime of the contracts. The rules that do not smooth the sharing rate over the entire lifetime of the contract are:
- one third (33%) for the landowner, two thirds (66%) for the the farmer in the first period and half and half (50% ; 50%) in the second period,
- one hundred percent (100%) for the farmer in the first period and half and half (50%; 50%)in the second period,
The rules that smooth out the sharing rate over the entire lifetime of the contract are:
- the half part sharing-rate (50% ; 50%) over the lifetime of the contract,
- the third part sharing-rate (1/3 ; 2/3) over the lifetime of the contract, with one third (33%) for the landowner (the transferor) and two thirds (66%) for the operator (the farmer)
The following proposition summarizes our findings in relation to the original research objective:

Proposition 4: The half part sharing-rate over the lifetime of the contract and the third part sharing-rate over the lifetime of the contract are the ones very close to the optimal long-term agricultural contracts.

4. Concluding Remarks

In this article, we've analyzed the optimality properties of "planted-shared" agricultural contracts. This issue is all the more important as these contracts, covering long cycle crops, must take into account the willingness of landowners to maintain incentives for farmers' efforts throughout the lifetime of the contract. Using a dynamic contract model with bilateral commitment based on the principal-agent paradigm, we have shown that in the case of the lack of an agricultural credit market, the optimal long-term contract is a contract with memory, essentially responding to a need for intertemporal smoothing of production sharing. This intertemporal smoothing effect is captured by the highlighted martingale property. Specifically, we have shown that the martingal property of the production sharing index is not only a necessary but also a sufficient condition for the optimality of the long-term "planted-shared" type agricultural contract. That's why, the half part sharing-rate over the lifetime of the contract and the third part sharing-rate over the lifetime of the contract are the ones very close to the optimal long-term agricultural contracts.

Facing the high demand of securing "planted-shared" contracts, public authorities could respond to this concern by legal formalization and public validation of such optimal contracts. These optimal contracts would enhance the relationships between land rights owners and non-owners for a more organized and equitable rural land market. There is a limitation in the model we developed in this paper. The issue of fairness in the optimal long-term contract was not of concern to us. However, the analysis resulted in two optimal sharing rules, namely: the third part sharing rate over the lifetime of the contract and the half part sharing rate over the lifetime of the contract. The study should on continue on the search for the most equitable rule.

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Appendix 1: *P'* program resolution

We consider the system of equations below which characterizes the optimality of the program (P'):

$$\begin{cases} P_1^i h'(\bar{u}_1^D) = \lambda + \mu P_1^i & (a) \\ (1 - P_1^i) h'(\underline{u}_1^D) = -\lambda + \mu(1 - P_1^i) & (b) \\ P_1^i C_2^{SB'}\left(u_2^D(\bar{y})\right) = \lambda + \mu P_1^i & (c) \\ (1 - P_1^i) C_2^{SB'}\left(u_2^D(\underline{y})\right) = -\lambda + \mu(1 - P_1^i) & (d) \end{cases}$$

(a) + (b) gives $\mu = P_1^i h'(\bar{u}_1^D) + (1 - P_1^i) h'(\underline{u}_1^D) > 0 \Rightarrow$ the intertemporal participation constraint is necessarily saturated. We obtain: $\lambda = P_1^i(1 - P_1^i)\left(h'(\bar{u}_1^D) - h'(\underline{u}_1^D)\right)$.

From (c) and (d), we have: $\lambda = P_1^i(1 - P_1^i)\left(C_2^{SB'}\left(u_2^D(\bar{y})\right) - C_2^{SB'}\left(u_2^D(\underline{y})\right)\right)$.

$$h'(\bar{u}_1^D) - h'(\underline{u}_1^D) = C_2^{SB'}\left(u_2^D(\bar{y})\right) - C_2^{SB'}\left(u_2^D(\underline{y})\right) \qquad (e)$$

with (a) and (c), we have: $h'(\bar{u}_1^D) = C_2^{SB'}\left(u_2^D(\bar{y})\right) \qquad (f)$

with (b) and (d), we have: $h'(\underline{u}_1^D) = C_2^{SB'}\left(u_2^D(\underline{y})\right) \qquad (g)$

however, $C_2^{SB}\left(u_2(\bar{y})\right) = P_1^i h\left(\bar{u}_2^D(\bar{y})\right) + (1 - P_1^i) h\left(\underline{u}_2^D(\bar{y})\right)$

$\Rightarrow C_2^{SB\prime}\left(u_2(\bar{y})\right) = P_1^i h'\left(\bar{u}_2^D(\bar{y})\right) + (1 - P_1^i) h'\left(\underline{u}_2^D(\bar{y})\right) \quad (h)$

and $C_2^{SB\prime}\left(u_2(\underline{y})\right) = P_1^i h'\left(\bar{u}_2^D(\underline{y})\right) + (1 - P_1^i) h'\left(\underline{u}_2^D(\underline{y})\right) \quad (i)$

when inserting (h) into (f) and (i) into (g), we have respectively:

$$h'(\bar{u}_1^D) = P_1^i h'\left(\bar{u}_2^D(\bar{y})\right) + (1 - P_1^i) h'\left(\underline{u}_2^D(\bar{y})\right)$$
$$h'(\underline{u}_1^D) = P_1^i h'\left(\bar{u}_2^D(\underline{y})\right) + (1 - P_1^i) h'\left(\underline{u}_2^D(\underline{y})\right)$$

These two equations satisfy the martingale property; they are simplified to:

$h'(u_1^D(y_1)) = E_{\tilde{y}_2}\left(h'(\tilde{u}_2^D(y_1))\right)$ for all $y_1 \in \{\underline{y}, \bar{y}\}$ QED

Appendix 2: Demonstration of the martingale property on the production sharing index

$h(u_1^D(y_1)) = \tau^i(y_1) = a^i y_1 + b$ and $h\left(\tilde{u}_2^D(y_1)\right) = \tilde{\tau}^{ij}(y_2) = \tilde{a}^{ij} y_2 + b$ (linear contract assumption); we can formulate the martingale solution:

$$h'_y(u_t^D(v_t f(x_{t-1}, e_t))) = h'_u(u_t^D(v_t f(x_{t-1}, e_t))) \times u'_y(y_t)$$

For simplicity, we'll write: $h'_y = h'_u \times u'_\tau \times \tau'_y$ given that: $\tau'_y = a$

$\Rightarrow h'_u = h'_y / a.u'_y$. we have : $u\left(\tau_1(y_1)\right) = u\left(a^i y_1 + b\right)$ and so,

$$h'_u(u_1^D(y_1)) = \frac{h'_y(u_1^D(y_1))}{a^i.u'_y}$$
$$h'_u\left(\tilde{u}_2^D(y_1)\right) = \frac{h'_y\left(\tilde{u}_2^D(y_1)\right)}{\tilde{a}^{ij}.u'_y}$$

The martingale solution is then written:

$$\frac{h'_y(u_1^D(y_1))}{a^i.u'_y} = E_{\tilde{y}_2}\left(\frac{h'_y\left(\tilde{u}_2^D(y_1)\right)}{\tilde{a}^{ij}.u'_y}\right)$$

$$\frac{1}{a^i} = E_{\tilde{y}_2}\left(\frac{h_y'\left(\tilde{u}_2^D(y_1)\right)}{h_y'(u_1^D(y_1))}\frac{1}{\tilde{a}^{ij}}\right)$$

Postulating the independence of the two terms under mathematical expectation, we can write,

$$E_{\tilde{y}_2}\left(\frac{h_y'\left(\tilde{u}_2^D(y_1)\right)}{h_y'(u_1^D(y_1))}\frac{1}{\tilde{a}^{ij}}\right) = E_{\tilde{y}_2}\left(\frac{1}{\tilde{a}^{ij}}\right) \times E_{\tilde{y}_2}\left[\frac{h_y'\left(\tilde{u}_2^D(y_1)\right)}{h_y'(u_1^D(y_1))}\right]$$

Finally, the martingale solution is

$$\frac{1}{a^i} = E_{\tilde{y}_2}\left(\frac{1}{\tilde{a}^{ij}}\right) \times E_{\tilde{y}_2}\left[\frac{h_y'\left(\tilde{u}_2^D(y_1)\right)}{h_y'(u_1^D(y_1))}\right]$$

Consider the following notations: $A^i = \frac{1}{a^i}$; $A^{ij} = \frac{1}{\tilde{a}^{ij}}$; $M \equiv E_{\tilde{y}_2}\left[\frac{h_y'\left(\tilde{u}_2^D(y_1)\right)}{h_y'(u_1^D(y_1))}\right]$.

If a^i and $\tilde{a}^{ij}$ are sharing rates, then A^i and A^{ij} are the inverse of the sharing rates. We call sharing indexes. With $M \gtreqless 1$, we have :

$$A^i \gtreqless E_{\tilde{y}_2}\left(A^{ij}\right)$$

If $M = 1$, we have $A^i = E_{\tilde{y}_2}\left(A^{ij}\right)$: the optimal sharing index is a martingale.
If $M > 1$, we have $A^i > E_{\tilde{y}_2}\left(A^{ij}\right)$: the optimal sharing index is an over martingale.
If $M < 1$, we have $A^i < E_{\tilde{y}_2}\left(A^{ij}\right)$: the optimal sharing index is a sub-martingale.